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A Comment on General Formulae for Polarization Observables in Deuteron Electrodisintegration and Linear Relations

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Abstract

We establish a simple, explicit relation between the formalisms employed in the treatments of polarization observables in deuteron two-body electrodisintegration published by Arenhövel, Leidemann, and Tomusiak in Few-Body Systems **15**, 109 (1993) and the results of the present authors published in Phys. Rev. C **40**, 2479 (1989). We comment on the overlap between the two sets of results.

In a recent issue of this journal an article [1] by Arenhövel, Leidemann and Tomusiak (ALT) on "General Formulae for Polarization Observables in Deuteron Electrodisintegration and Linear Relations" has appeared. Four years earlier [2] we published a comprehensive treatment of polarization observables in this reaction (DG), and since the ALT paper makes no reference to our work we feel obliged to comment on these two papers, and to discuss the relationship between these two approaches. In this comment we will establish a simple and explicit relation between the transition amplitudes in the two approaches, whereupon all of our results [2] become immediately applicable to the ALT formalism.

Before we compare these two papers in detail, we review the arguments which determine the number of real observables which can be measured in deuteron electrodisintegration. The total number of spin variables in this reaction are $3 \times 3 \times 2 \times 2 = 36$, but because of the parity constraint, only half of these complex amplitudes, 18, are independent. The number of real bilinear products which can be formed from these 18 complex amplitudes is $18 \times 18 = 324$ (since A^*B and B^*A are equivalent to two real functions). However, since there are "only" 18 independent complex amplitudes, and since the overall phase can never be determined, all of these 324 observables depend on products of only 35 independent real functions. The problem of completely measuring deuteron electrodisintegration reduces to the problem of designing a program of measurements from which the 35 independent real functions can unambigously extracted from combinations of the 324 bilinear products measured in actual experiments. Clearly not *all* possible measurements are needed for a complete determination, and as more and more measurements are added to the data base, greater and greater care must be taken to find new measurements which give truly independent information.

Because of this redundancy, in DG we discussed all possible spin observables which can be measured in the reaction $d(e, e'N_1)N_2$, where nucleon N_2 is not observed, and therefore its polarization is not detected. Hence we limited ourselves to observables in which the polarization of the virtual photon, the deuteron target, and *one* outgoing nucleon are measured, either singlely or in all possible combinations. Choosing a hybrid transversity basis we were able to obtain a comparatively simple result, and demonstrated that 162 bilinear products of amplitudes can be measured by looking at reactions where $N_1 = p$. Adding the cases where $N_1 = n$ gives another 162 bilinear products of amplitudes, but only 80 of these are new (see below). We did not discuss measurements in which the polarization of both of the outgoing nucleons are measured (which requires $d(e, e'\vec{n}\,\vec{p})$ measurements) but the remaining 82 products could be measured in this way. We also showed that the 162 observables accessible to $N_1 = p$ measurements could *not* completely determine the 35 independent real quantities, *even though 162 is far greater that 35.* At least one neutron polarization measurement must be made before all 35 independent real quantities can be extracted, but one such additional measurement is sufficient, in principle, to complete the program.

The ALT paper is an extension of an earlier paper [3] on complete classification of all polarization experiments in deuteron photodisintegration. They extend their photodisintegration formalism to include longitudinal polarization of the virtual photon, and the number of amplitudes is accordingly increased from 12 to 18. They discuss *all* possible polarization measurements, including those which can be obtained from $d(e, e' \vec{n}\, \vec{p})$, and hence should obtain all of the 324 bilinear products. However, their formalism generates twice this many (648), and they spend some time showing how the parity constraint generates the necessary 324 linear relations between these 648 bilinear products, all of which are nonzero in their formalism. Unfortunately, the linear relations between the 648 amplitudes make it difficult to see which measurements are sufficient to extract the 35 truly independent real functions needed to completely determine all deuteron electrodisintegration observables. At the end of their paper they say that, in order to fully determine all observables in deuteron electrodisintegration, "one cannot totally avoid" measuring two observables in which the polarizations of *both* outgoing nucleons are measured. This conclusion contradicts the result of DG, where we showed that *no* such measurements are *required* (although it might turn out that a particular separation strategy might make use of such measurements). We will compare the results of these two papers in more detail shortly.

Before turning to the details of this discussion, it might help the reader to look at these two papers within the historical context. The ALT paper is the latest in a long series [3–5] that can be traced back to J.J. De Swart's founding paper from 1959 [6], which relied on the nonrelativistic spin polarization formalism developed by Ashkin and Wolfenstein [7].

Our approach has a similarly long lineage dating back to the relativistic spin polarization (helicity) formalism of Jacob and Wick (JW) [8] who, among other things, established a link with the nonrelativistic formalisms. This method allows a simple exploration of parity and other symmetries, as well as complete separations of amplitudes. The first application of the JW formalism to the problem of deuteron photo- and electrodisintegration was made by LeBellac, Renard and Tran Thanh Van in a series [9–13] of formal and practical papers in the mid-sixties. At that time the helicity formalism was still sufficiently new to warrant a comparison with the older nonrelativistic formalism. The relation between the two, in the form of formulae for multipoles, was explicitly spelled out in Appendix B of Ref. [9] for the photodisintegration amplitudes and in section 4.2 of Ref. [11], as well as in section 5 of Ref. [12] for the electrodisintegration amplitudes. Although the helicity formalism has its complexities, it does lead to final results with patterns simple enough to allow for a comparatively simple discussion of separation strategies.

We now turn to a detailed comparison of the ALT and DG papers. Our approach (DG) begins with the use of helicity amplitudes with parity transformation properties summarized by the following relation

$$\langle \lambda_p \; \lambda_n | J_{\lambda_\gamma} | \lambda_D \rangle \equiv \langle \lambda_p \; \lambda_n | J \cdot \epsilon_{\lambda_\gamma} | \lambda_D \rangle \tag{1a}$$

$$\langle \lambda_p \; \lambda_n | J_{\lambda_\gamma} | \lambda_D \rangle = (-1)^{(\lambda_p - \lambda_n) - (\lambda_\gamma - \lambda_D)} \langle -\lambda_p \; -\lambda_n | J_{-\lambda_\gamma} | -\lambda_D \rangle \; , \tag{1b}$$

where $J \cdot \epsilon_\lambda = J_\mu \, \epsilon^\mu_\lambda$, using the Bjorken and Drell metric [14], and the initial state consists of a virtual photon with helicity λ_γ and a deuteron (particle No. 2 in the sense of Jacob and Wick [8]) with helicity λ_D, and the final state consists of an outgoing proton with helicity λ_p and neutron (particle No. 2) with helicity λ_n. The hadronic response current, J^μ, is defined in the ejectile plane, defined in Fig. 2 of Ref. [2]. The ALT paper is based on the use of reduced amplitudes $t_{sm_s\lambda m}$, where λ and m are the virtual photon and deuteron spin projections in the direction of the momentum transferred by the scattered electron, $\mathbf{q}$, and the spins of the outgoing nucleons are coupled into states of total spin $s = 0$ or 1, with total spin projection m_s in the direction of the relative momentum $\mathbf{p}_{np}$ of the outgoing np pair in the center of momentum (c.m.) frame. For our present task it is a fortunate coincidence that ALT chose the spin quantization axis for the deuteron to be in the direction of $\mathbf{q}$, and the quantization for the final state nucleon spins to be along the direction of the relative np momentum in the c.m. frame of the outgoing pair [1], because this makes it easy to identify their spin projections with our helicities, as follows:

$$\begin{aligned}\lambda &= \lambda_\gamma \\ m &= -\lambda_D \\ m_s &= \lambda_p - \lambda_n \,. \end{aligned} \tag{2}$$

However, the ALT decision to work with amplitudes with a definite value of the total nuclear spin s leads to subsequent differences in appearance between the two approaches. In spite of this, the simple relations (2) allow us to connect our helicity formalism with the ALT formalism using only Clebsch-Gordan coefficients.

For $s = 1, m_s = \pm 1$, the relation to the helicity states is straightforward:

$$t_{11\lambda m} = C\langle +\tfrac{1}{2} \;\; -\tfrac{1}{2}|J_\lambda| - m\rangle \tag{3a}$$

$$t_{1-1\lambda m} = C\langle -\tfrac{1}{2} \;\; +\tfrac{1}{2}|J_\lambda| - m\rangle \;, \tag{3b}$$

where C is a proportionality factor. The Jacob-Wick (helicity) parity conservation relation (1b) gives the following parity relations for the $m_s = \pm 1$ ALT amplitudes

$$\begin{aligned} t_{1\pm1\lambda m} &= C\langle \pm\tfrac{1}{2} \;\; \mp\tfrac{1}{2}|J_\lambda| - m\rangle \\ &= (-1)^{m_s+\lambda+m} C\langle \mp\tfrac{1}{2} \;\; \pm\tfrac{1}{2}|J_{-\lambda}|m\rangle \\ &= (-1)^{m_s+\lambda+m} t_{1\,\mp1\,-\lambda\,-m} \\ &= (-1)^{1+s+m_s+\lambda+m} t_{1\,\mp1\,-\lambda\,-m} \,, \end{aligned} \tag{4}$$

in agreement with Eq. (4) of ALT.

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[1]This choice was made in the original treatment [6], but was forgotten in the meantime and that has lead to some confusion. Compare the final state polarization results in Refs. [4] and [5].

The comparison of the $s = 1, m_s = 0$ and $s = 0, m_s = 0$ states is less straightforward. In these cases we need to form the symmetric and antisymmetric normalized linear combinations of the two outgoing nucleon helicities and find:

$$t_{10\lambda m} = C\frac{1}{\sqrt{2}}\left[\langle +\tfrac{1}{2}\ +\tfrac{1}{2}|J_\lambda| - m\rangle + \langle -\tfrac{1}{2}\ -\tfrac{1}{2}|J_\lambda| - m\rangle\right] \tag{5a}$$

$$t_{00\lambda m} = C\frac{1}{\sqrt{2}}\left[\langle +\tfrac{1}{2}\ +\tfrac{1}{2}|J_\lambda| - m\rangle - \langle -\tfrac{1}{2}\ -\tfrac{1}{2}|J_\lambda| - m\rangle\right] \quad . \tag{5b}$$

The symmetric combination Eq. (5a) is actually the $s = 1$ amplitude because it has the appropriate phase under the parity transformation as the rest of the triplet:

$$\begin{aligned} t_{10\lambda m} &= C\frac{1}{\sqrt{2}}\left[\langle +\tfrac{1}{2}\ +\tfrac{1}{2}|J_\lambda| - m\rangle + \langle -\tfrac{1}{2}\ -\tfrac{1}{2}|J_\lambda| - m\rangle\right] \\ &= C\frac{1}{\sqrt{2}}\left[(-1)^{\lambda+m}\langle -\tfrac{1}{2}\ -\tfrac{1}{2}|J_{-\lambda}|m\rangle + (-1)^{\lambda+m}\langle +\tfrac{1}{2}\ +\tfrac{1}{2}|J_{-\lambda}|m\rangle\right] \\ &= (-1)^{\lambda+m} t_{10\,-\lambda\,-m} \\ &= (-1)^{m_s+\lambda+m} t_{10\,-\lambda\,-m} \\ &= (-1)^{1+s+m_s+\lambda+m} t_{10\,-\lambda\,-m} \,. \end{aligned} \tag{6}$$

The singlet Eq. (5b), on the other hand, is antisymmetric and hence has the opposite phase under parity:

$$\begin{aligned} t_{00\lambda m} &= C\frac{1}{\sqrt{2}}\left[\langle +\tfrac{1}{2}\ +\tfrac{1}{2}|J_\lambda| - m\rangle - \langle -\tfrac{1}{2}\ -\tfrac{1}{2}|J_\lambda| - m\rangle\right] \\ &= C\frac{1}{\sqrt{2}}\left[(-1)^{\lambda+m}\langle -\tfrac{1}{2}\ -\tfrac{1}{2}|J_{-\lambda}|m\rangle - (-1)^{\lambda+m}\langle +\tfrac{1}{2}\ +\tfrac{1}{2}|J_{-\lambda}|m\rangle\right] \\ &= (-1)^{1+\lambda+m} t_{00\,-\lambda\,-m} \\ &= (-1)^{1+m_s+\lambda+m} t_{00\,-\lambda\,-m} \\ &= (-1)^{1+s+m_s+\lambda+m} t_{00\,-\lambda\,-m} \quad . \end{aligned} \tag{7}$$

All cases can be described by a single formula

$$t_{sm_s\lambda m} = (-1)^{1+s+m_s+\lambda+m} t_{s\,-m_s\,-\lambda\,-m} \ , \tag{8}$$

which is exactly Eq. (4) of [1]. Thus we have shown that the ALT amplitudes are simple linear combinations of the helicity amplitudes. From this point on the comparison between the two papers is strictly a matter of transcription of the results from one notation to the other.

Of course, in matters like this, transparent and concise notation is very important, and following a suggestion by Moravcsik, we found that a hybrid transversity basis gave comparatively simple results. Our hybrid basis is obtained from the helicity basis by rotating the amplitudes by $-\pi/2$ around the x axis. This is equivalent to using amplitudes in which hadron spins are quantized with respect to the y axis, but the photon spin remains quantized with respect to the z axis. The construction of this basis is described in Sec. II.E of DG, and the explicit transformations for helicity basis to hybrid basis are given in the Appendix of that article. This basis could also be expressed in terms of the ALT amplitudes by combining the relations (3) and (5) with the transformations given in DG. Final results for the observables in which $N_1 = p$, expressed as bilinear products of the hybrid amplitudes, are summarized in Tables X–XII of DG. Because of the use of the hybrid basis, one-half of the entries in these tables are zero, and *all* of the 162 nonzero entries are linear combinations of 162 *different* real bilinear products. In the language of ALT, there are *no* linear relations connecting these products to each other.

As discussed in DG, it is possible to obtain a simple understanding of the origin of the 162 independent real bilinear products determined by reactions in which $N_1 = p$. These measurements divide the 18 independent complex amplitudes into two disjoint classes of 9 amplitudes each [defined explicitly in Eq. (99) of DG], in the sense that these measurements determine all products of amplitudes in each class, but no products of amplitudes in one class with those in another. Hence $N_1 = p$ measurements determine $9 \times 9 + 9 \times 9 = 162$ independent products. Now, the same Tables X–XII can be used to obtain the observables for neutron measurements (in which $N_1 = n$), provided one *exchanges five of the amplitudes in one class with five in the other* (see DG for details). Hence neutron measurements determine $2 \times (4 \times 5 + 5 \times 4) = 80$ new real bilinear products, the remaining 82 being identical to those already fixed by the proton measurements. In the ALT language, the identity of the 82 products which occur in both proton and neutron measurements could be written as linear relations, but the simplicity of the pattern of results given in Tables X–XII makes this unnecessary. Finally, the remaining $2 \times (4 \times 4 + 5 \times 5) = 82$ products of amplitudes arising from products amplitudes with one from each of the two groups of 5 amplitudes exchanged in the $p \to n$ substitution, and similar products between the two groups of 4 amplitudes not exchanged in the $p \to n$ substitution, cannot be determined by either class of experiments, and require $d(e, e'\vec{p}\vec{n})$ experiments, as stated at the beginning of this comment. That does *not* mean, however, that these experiments are necessary for the complete separation of amplitudes.

We conclude by emphasizing that any attempt to discuss complete separations in reactions as complex as deuteron electrodisintegration, or to find "the most suitable complete set" of amplitudes, requires that the relations between the observables and the bilinear products from which they are determined be as simple as possible. We believe that the hybrid transversity basis, popularized by Moravscik and developed in DG, is just such a basis. The ALT choice of the final state spin quantization axis makes a simple, direct link between their amplitudes and the hybrid basis possible. Despite the fact that one can view the problem of complete separation of amplitudes as solved by this basis, it would still be satisfying to see the ALT results expressed in this basis.

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